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Necessary and Sufficient Conditions for Oscillations of Neutral Differential Equations

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Consider the neutral delay differential equation

$$x'(t) + px'(t-\tau) + qx(t-\sigma) = 0, \qquad t \geqq t_0, \tag{$*$}$$

where τ, q, and σ are positive constants while p is a real parameter. Then a necessary and sufficient condition for all solutions of $(*)$ to oscillate is that the characteristic equation

$$\lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0$$

of $(*)$ has no real roots. © 1987 Academic Press, Inc.

1. INTRODUCTION

Neutral delay differential equations (NDDE) are differential equations in which the highest order derivative of the unknown function appears with the argument t (present state) as well as one or more retarded arguments (past histories).

The oscillation theory of delay differential equations has been extensively developed during the past few years. See, for example, [1, 13–16, 19–24] and the references cited therein. It is to be noted, however, that the literature is scarce concerning the oscillatory behavior of solutions of neutral differential equations. Only very recently Ladas, Sficas, Grammatikopoulos, Grove, and Meimaridou, see [7–10, 17, 18], studied the oscillatory and asymptotic behavior of solutions of neutral equations, which to the best of our knowledge are the only papers at the present time dealing with this problem. For some results about second order nonlinear neutral equations see [27]. However, the results obtained in those papers [7–10, 17, 18, 27] lead to sufficient conditions only.

Our aim in this paper is to obtain a necessary and sufficient condition

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under which all solutions of the first order neutral delay differential equation

$$x'(t)+px'(t-\tau)+qx(t-\sigma)=0, \qquad t \geqq t_0, \tag{1}$$

oscilllate, where τ, q, σ are positive constants and p is a real parameter. Indeed we prove that every solution of Eq. (1) oscillates if and only if its characteristic equation

$$\lambda+\lambda pe^{-\lambda\tau}+qe^{-\lambda\sigma}=0$$

has no real roots. That is, the oscillatory character of the solutions is determined by the roots of the characteristic equation. It is to be noted, however, that in general the behavior of solutions of neutral delay differential equations exhibit features which are not true for nonneutral delay equations. There are examples (see [3, 4, 11, 12, 25, 26]) of neutral differential equations with all the characteristic roots in the negative half-plane or with all the characteristic roots simple and on the imaginary axis and yet the equation has unbounded solutions.

The problem of asymptotic and oscillatory behavior of solutions of neutral delay differential equations is of both theoretical and practical interest. Note that equations of this type appear in networks containing lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits (see [3, 25]).

Let $\varphi \in C[t_0-\hat{\tau}, t_0], \mathbb{R})$, where $\hat{\tau}=\max\{\tau, \sigma\}$. By a solution of Eq. (1) with initial function φ at t_0, we mean a function $x \in C([t_0-\hat{\tau}, \infty], \mathbb{R})$ such that $x(t)=\varphi(t)$ for $t_0-\hat{\tau} \leqq t \leqq t_0$, $x(t)+px(t-\tau)$ is continuously differentiable, and x satisfies (1) for all $t \geqq t_0$. Using the method of steps, it follows that for every continuous function φ, there exists a unique solution of Eq. (1) valid for $t \geqq t_0$. For further questions on existence, uniqueness and continuous dependence, see Driver [5, 6], Bellman and Cooke [2], and Hale [12].

As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

2. Main Result

Our aim result is the following:

THEOREM. *Consider the neutral delay differential equation*

$$x'(t)+px'(t-\tau)+qx(t-\sigma)=0, \qquad t \geqq t_0 \tag{1}$$

where τ, q, σ are positive constants and p is a real parameter. Then a necessary and sufficient condition for all solutions of (1) *to oscillate is that the characteristic equation*

$$\lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0 \tag{2}$$

of (1) *has no real roots.*

Observe that for $p = 0$, Eq. (1) leads to

$$x'(t) + qx(t-\sigma) = 0, \tag{3}$$

and

$$q\sigma > \frac{1}{e} \tag{4}$$

is a necessary and sufficient condition for all solutions of (3) to oscillate (see [16]). Note that in this case the characteristic equation is

$$\lambda + qe^{-\lambda\sigma} = 0 \tag{5}$$

and (4) is equivalent to the fact that (5) has no real roots. Also for $p = -1$ Eq. (1) leads to

$$x'(t) - x'(t-\tau) + qx(t-\sigma) = 0 \tag{6}$$

and every solution of (6) oscillates (see [17]).

From these observations it is clear that to prove the theorem we have to consider the following three cases for p:

(i) $-1 < p < 0$,

(ii) $p > 0$, and

(iii) $p < -1$.

The following lemma indicates how from an eventually positive solution $x(t)$ of (1) we can construct auxiliary solutions with some nice asymptotic properties. The proof of this lemma is omitted since it can be easily derived combining Lemma 1 and Theorem 5 in [17] and Theorems 9 and 6 in [7].

LEMMA. *Let $x(t)$ be an eventually positive solution of* (1). *Then*

(a) *for $p > -1$ the function $z(t) = x(t) + px(t-\tau)$ is an eventually positive solution or* (1), *decreasing and* $\lim_{t\to\infty} z(t) = 0$.

(b) *for $p < -1$ the function $z(t) = -x(t) - px(t-\tau)$ is an eventually positive solution of* (1), *increasing and* $\lim_{t\to\infty} z(t) = +\infty$.

Proof of the Theorem. The theorem will be proved in the contrapositive form: There is a nonoscillatory solution of (1) if and only if the characteristic equation (2) has a real root. Assume first that (2) has a real root. Then (1) has the nonoscillatory solution $x(t)=e^{\lambda t}$.

Assume, conversely, that there is a nonoscillatory solution $x(t)$ of (1) which, without loss of generality, can be considered, eventually positive.

Consider now the following cases:

(i) *The case* $-1<p<0$.

Set

$$z(t)=x(t)+px(t-\tau). \tag{7}$$

Then, by the lemma, $z(t)$ is eventually positive and decreasing and without loss of generality $x(t)$ can be considered also decreasing. It is easy to see that $z(t)<x(t-\sigma)$. Define the set

$$\Lambda(z)=\{\lambda>0: z'(t)+\lambda z(t)<0 \text{ eventually}\}. \tag{8}$$

From (1) we have

$$0=z'(t)+qx(t-\sigma)>z'(t)+qz(t), \qquad \text{eventually,}$$

so that $q\in\Lambda(z)$. That is, $\Lambda(z)$ is nonempty.

On the other hand, it follows that

$$0=z'(t)+qx(t-\sigma)=z'(t)+qz(t-\sigma)-pqx(t-\tau-\sigma),$$

and integrating from t to $t+\tau$ we find

$$z(t+\tau)-z(t)+q\int_t^{t+\tau} z(s-\sigma)\,ds-pq\int_t^{t+\tau} x(s-\tau-\sigma)\,ds=0.$$

Taking into account that both $z(t)$ and $x(t)$ are positive and decreasing, the last equation yields

$$z(t+\tau)-z(t)+q\tau z(t+\tau-\sigma)-pq\tau x(t-\sigma)\leqq 0,$$

which implies that

$$-pq\tau x(t-\sigma)\leqq z(t).$$

Thus

$$0=z'(t)+qx(t-\sigma)\leqq z'(t)+\left(\frac{1}{-p\tau}\right)z(t), \qquad \text{eventually.}$$

Therefore $\lambda_0 = 1/-p\tau$ is an upper bound of $\Lambda(z)$ which does not depend on z. Thus $\Lambda(z)$ is nonempty and bounded from above.

Let $\lambda \in \Lambda(z)$ and consider the function

$$w(t) \equiv Tz = z(t) + pz(t-\tau). \tag{9}$$

Set

$$m = \inf_{\lambda > 0} \{-\lambda - p\lambda e^{\lambda\tau} + qe^{\lambda\sigma}\}. \tag{10}$$

Assume, for the sake of contradiction, that (2) has no real roots. Furthermore $\lim_{\lambda \to \infty} (-\lambda - p\lambda e^{\lambda\tau} + qe^{\lambda\sigma}) = +\infty$ and therefore m is positive. We will show that $\lambda + m \in \Lambda(w)$. From (1) and (9) and the fact that both $z(t)$ and $w(t)$ are solutions of (1) we obtain

$$w'(t) = -qz(t-\sigma). \tag{11}$$

Define $\varphi(t) = e^{\lambda t}z(t)$. *Then*

$$\varphi'(t) = [z'(t) + \lambda z(t)]\, e^{\lambda t} < 0, \qquad \text{eventually}$$

and therefore φ is eventually decreasing. Since $z(t) = e^{-\lambda t}\varphi(t)$, (9) and (11) give, respectively,

$$w(t) = e^{-\lambda t}\varphi(t) + pe^{-\lambda t}e^{\lambda\tau}\varphi(t-\tau)$$

and

$$w'(t) = -qe^{-\lambda t}e^{\lambda\sigma}\varphi(t-\sigma).$$

Now using the fact that $\varphi(t)$ is decreasing, we obtain

$$\begin{aligned} w'(t) + (\lambda + m)\, w(t) &= e^{-\lambda t}[-qe^{\lambda\sigma}\varphi(t-\sigma) + (\lambda+m)\,\varphi(t) \\ &\quad + (\lambda+m)\, pe^{\lambda\tau}\varphi(t-\tau)] \\ &< e^{-\lambda t}\varphi(t)[-qe^{\lambda\sigma} + \lambda + m + \lambda pe^{\lambda\tau} + mpe^{\lambda\tau}] \\ &< e^{-\lambda t}\varphi(t)[\lambda + p\lambda e^{\lambda\tau} - qe^{\lambda\sigma} + m] \\ &\leqq e^{-\lambda t}\varphi(t)[-m+m] = 0 \end{aligned} \tag{12}$$

which implies that $\lambda + m \in \Lambda(w)$. Now set

$$z \equiv z_0,\ w \equiv Tz_0 = z_1, \quad z_2 = Tz_1 \quad \text{and in general} \quad z_n = Tz_{n-1}, \quad n = 1, 2, \ldots,$$

and observe that for $\lambda \in \Lambda(z) \equiv \Lambda(z_0) \Rightarrow \lambda + nm \in \Lambda(z_n)$, $n = 1, 2, \ldots$, which is a contradiction since λ_0 is a common upper bound for all $\Lambda(z_n)$.

(ii) *The case* $p>0$.

As in case (i) we set

$$z(t)=x(t)+px(t-\tau) \tag{7}$$

and

$$\Lambda(z)=\{\lambda>0\colon z'(t)+\lambda z(t)<0 \text{ eventually}\} \tag{8}$$

In this case using the characteristic equation of (1)

$$F(\lambda)=\lambda+\lambda pe^{-\lambda\tau}+qe^{-\lambda\sigma}=0$$

we see that if $\tau \geqq \sigma > 0$, $F(0)=q>0$ while $\lim_{\lambda\to-\infty} F(\lambda)=-\infty$ and therefore (1) always has nonosillatory solutions. Thus $\sigma>\tau$ is a necessary condition for all solutions of (1) to oscillate (see also [7]). By the lemma, $z(t)$ is eventually positive and decreasing and without loss of generality $x(t)$ can be considered also decreasing. Thus

$$z(t)=x(t)+px(t-\tau)<x(t-\sigma)+px(t-\sigma)=(1+p)\,x(t-\sigma)$$

That is,

$$x(t-\sigma)>\frac{1}{1+p}z(t).$$

From (1) we have

$$0=z'(t)+qx(t-\sigma)>z'(t)+\frac{q}{1+p}z(t), \qquad \text{eventually,}$$

so that $q/(1+p)\in\Lambda(z)$. That is, $\Lambda(z)$ is nonempty.

Now we will show that $\Lambda(z)$ is bounded from above. Observe that Eq. (1) is autonomous and $z(t)$, given by (7), as a linear combinaton of solutions of (1) is itself a solution of (1) and therefore

$$z'(t)+pz'(t-\tau)+qz(t-\sigma)=0.$$

Also $z'(t)=-qx(t-\sigma)<0$ and $z''(t)=-qx'(t-\sigma)>0$, that is, $z'(t)$ is increasing and therefore $z'(t)>z'(t-\tau)$. Thus the last equation yields

$$(1+p)\,z'(t-\tau)+qz(t-\sigma)\leqq 0$$

or

$$z'(t)+\frac{q}{1+q}z(t-(\sigma-\tau))\leqq 0.$$

Integrating the last inequality first from $t-((\sigma-\tau)/2)$ to t and then from t to $t+((\sigma-\tau)/2)$ (see also [19, Lemma]), we have

$$z(t-(\sigma-\tau)) \leqq \left[\frac{2(1+p)}{q(\sigma-\tau)}\right]^2 z(t). \tag{13}$$

Next integrating $z'(t)+qx(t-\sigma)=0$ over the interval $[t-\sigma+\tau, t]$, we obtain

$$z(t)-z(t-(\sigma-\tau))+q\int_{t-(\sigma-\tau)}^{t} x(s-\sigma)\,ds=0$$

and, since $x(t)$ is positive and decreasing and $z(t)$ is positive, we have

$$q(\sigma-\tau)\,x(t-\sigma) \leqq z(t-(\sigma-\tau)).) \tag{14}$$

Combining (13) and (14), we obtain

$$x(t-\sigma) \leqq \frac{[2(1+p)]^2}{q^3(\sigma-\tau)^3}\, z(t).$$

Thus

$$0=z'(t)+qx(t-\sigma) \leqq z'(t)+\left[\frac{4(1+p)^2}{q^2(\sigma-\tau)^3}\right] z(t), \qquad \text{eventually.}$$

Therefore $\lambda_0=4(1+p)^2/q^2(\sigma-\tau)^3$ is an upper bound of $\Lambda(z)$ which does not depend on z. Thus $\Lambda(z)$ is nonempty and bounded from above.

Now if we follow the same procedure as in the last part of the proof of case (i), considering $\lambda \in \Lambda(z)$ and defining $w(t)$, m as in (9) and (10), we will show that $\lambda+\mu \in \Lambda(w)$, where $\mu=m/(1+pe^{\lambda_0\tau})>0$. As in (12) we obtain

$$\begin{aligned} w'(t)&+(\lambda+\mu)\,w(t) \\ &= e^{-\lambda t}[-qe^{\lambda\sigma}\varphi(t-\sigma)+(\lambda+\mu)\,\varphi(t)+(\lambda+\mu)\,pe^{\lambda\tau}\varphi(t-\tau)] \end{aligned}$$

(and since $\varphi(t)$ is decreasing and $\sigma>\tau$)

$$\begin{aligned} &\langle e^{-\lambda t}\varphi(t-\sigma)[-qe^{\lambda\sigma}+\lambda+\mu+\lambda pe^{\lambda\tau}+\mu pe^{\lambda\tau}] \\ &=e^{-\lambda t}\varphi(t-\sigma)[\lambda+p\lambda e^{\lambda\tau}-qe^{\lambda\sigma}+\mu(1+pe^{\lambda\tau})] \\ &\leqq e^{-\lambda t}\varphi(t-\sigma)[-m+\mu(1+pe^{\lambda_0\tau})]=0 \end{aligned}$$

which implies that $\lambda+\mu \in \Lambda(w)$. This (as in case (i)) leads to a contradiction.

(iii) *The case* $p < -1$.

First, we assume $\tau \geqq \sigma$. Set

$$z(t) = -x(t) - px(t-\tau). \tag{15}$$

By the lemma, $z(t)$ is eventually positive and increasing and without loss of generality $x(t)$ can be considered also increasing. From (15), we have

$$-px(t-\tau) > z(t) \Rightarrow x(t-\tau) > \frac{1}{(-p)} z(t) \text{ or } x(t-\sigma) > \frac{1}{(-p)} z(t).$$

Define the set

$$\Lambda(z) = \{\lambda > 0: -z'(t) + \lambda z(t) < 0 \qquad \text{eventually}\}. \tag{16}$$

From (1)

$$0 = -z'(t) + qx(t-\sigma) > -z'(t) + \frac{q}{(-p)} z(t), \qquad \text{eventually}$$

so that $q/(-p) \in \Lambda(z)$. That is, $\Lambda(z)$ is nonempty.

Now we will show that $\Lambda(z)$ is bounded from above. From (15) we have

$$x(t-\tau) = \frac{1}{-p} [x(t) + z(t)]$$

and therefore

$$z'(t) = qx(t-\sigma) = \frac{q}{-p} [x(t+\tau-\sigma) + z(t+\tau-\sigma)].$$

Integrating the last equation from $t-\tau$ to t and taking into account that $x(t)$ and $z(t)$ are eventually positive and increasing, we obtain

$$z(t) - z(t-\tau) = \frac{q}{-p} \int_{t-\tau}^{t} [x(s+\tau-\sigma) + z(s+\tau-\sigma)]\, ds \geqq \frac{q}{-p} \tau x(t-\sigma)$$

or

$$x(t-\sigma) \leqq \frac{-p}{q\tau} z(t), \qquad \text{eventually.}$$

Thus

$$0 = -z'(t) + qx(t-\sigma) \leqq -z'(t) + \left(\frac{-p}{\tau}\right) z(t), \qquad \text{eventually.}$$

Therefore $\lambda_0 = (-p/\tau)$ is an upper bound of $\Lambda(z)$ which does not depend on z. Thus $\Lambda(z)$ is nonempty and bounded from above.

Next we follow a procedure analogous to that given in the last part of the proof of case (i). Let $\lambda \in \Lambda(z)$ and consider the function

$$w(t) = -z(t) - pz(t-\tau) \tag{17}$$

which is also a solution of (1) and therefore from (1) $w'(t) = qz(t-\sigma)$. Assume, for the sake of contradiction, that (2) has no real roots and set

$$m = \inf_{\lambda > 0} \{\lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma}\}. \tag{18}$$

Then m, is positive. We will show that $\lambda + \mu \in \Lambda(w)$ where $\mu = m/-pe^{-\lambda_0\tau} > 0$. Define $\varphi(t) = e^{-\lambda t}z(t)$. Then

$$\varphi'(t) = [z'(t) - \lambda z(t)]\, e^{-\lambda t} > 0, \qquad \text{eventually}$$

and therefore φ is eventually increasing. Since $z(t) = e^{\lambda t}\varphi(t)$, similarly as in (12), we obtain

$$\begin{aligned}
&-w'(t) + (\lambda+\mu)\, w(t)\\
&\quad = e^{\lambda t}[-qe^{-\lambda\sigma}\varphi(t-\sigma) - (\lambda+\mu)\,\varphi(t) + (\lambda+\mu)(-p)\, e^{-\lambda\tau}\varphi(t-\tau)]
\end{aligned}$$

(and since $\varphi(t)$ is increasing and $\tau \geqq \sigma$)

$$\begin{aligned}
&\langle e^{\lambda t}\varphi(t-\sigma)[-qe^{-\lambda\sigma} - (\lambda+\mu) + (\lambda+\mu)(-p)\, e^{-\lambda\tau}]\\
&= e^{\lambda t}\varphi(t-\sigma)[-\lambda - p\lambda e^{-\lambda\tau} - qe^{-\lambda\sigma} + \mu(-pe^{-\lambda\tau} - 1)]\\
&\leqq e^{\lambda t}\varphi(t-\sigma)[-m + \mu(-p)\, e^{-\lambda\tau} - \mu]\\
&\langle e^{\lambda t}\varphi(t-\sigma)[-m + \mu(-p)\, e^{-\lambda_0\tau}] = 0
\end{aligned}$$

which implies that $\lambda + \mu \in \Lambda(w)$ and, as before, we are led to a contradiction.

To complete the proof in this case that $p < -1$ we have to assume that $\sigma > \tau$. Here we set

$$z(t) = -x(t) - px(t-\tau) + q\int_{t-\sigma}^{t-\tau} x(s)\, ds \tag{19}$$

which is a solution of (1) eventually positive. From (1), we have

$$z'(t) = qx(t-\tau) > 0$$

which implies that $z(t)$ is increasing and without loss of generality $x(t)$ can be considered also increasing. Therefore (19) yields

$$z(t) < -px(t-\tau) + q\int_{t-\sigma}^{t-\tau} x(s)\,ds < -px(t-\tau) + q(\sigma-\tau)\,x(t-\tau)$$
$$= [q(\sigma-\tau)-p]\,x(t-\tau)$$

or

$$x(t-\tau) > \frac{1}{q(\sigma-\tau)-p}\,z(t).$$

As before, we define the set

$$\Lambda(z) = \{\lambda > 0\colon -z'(t) + \lambda z(t) < 0 \qquad \text{eventually}\}. \tag{16}$$

In view of the last inequality, we have

$$0 = -z'(t) + qx(t-\tau) > -z'(t) + \frac{q}{q(\sigma-\tau)-p}\,z(t), \qquad \text{eventually}$$

so that $q/(q(\sigma-\tau)-p) \in \Lambda(z)$. That is, $\Lambda(z)$ is nonempty.

Next, we will show that $\Lambda(z)$ is bounded from above. Since $z(t)$ is a solution of (1) it follows that

$$z'(t) + pz'(t-\tau) + qz(t-\sigma) = 0$$

which implies

$$z'(t) + pz'(t-\tau) \leqq 0.$$

But $z'(t) = qx(t-\tau)$ and thus

$$qx(t-\tau) + pz'(t-\tau) \leqq 0.$$

Integrating the last inequality in the interval $[t, t+\tau]$ and taking into account that $x(t)$ is positive and increasing and $z(t)$ eventually positive, we obtain

$$0 \geqq q\int_{t}^{t+\tau} x(s-\tau)\,ds + pz(t) - pz(t-\tau)$$
$$\geqq q\tau x(t-\tau) + pz(t) - pz(t-\tau)$$
$$> q\tau x(t-\tau) + pz(t)$$

or

$$x(t-\tau) \leqq \frac{(-p)}{q\tau} z(t), \qquad \text{eventually.}$$

Thus

$$0 = -z'(t) + qx(t-\tau) \leqq -z'(t) + \frac{(-p)}{\tau} z(t), \qquad \text{eventually,}$$

which implies that $\lambda_0 = (-p/\tau)$ is an upper bound of $\Lambda(z)$ which does not depend on z. Thus $\Lambda(z)$ is nonempty and bounded from above.

Now let $\lambda \in \Lambda(z)$ and consider the function [cf. (19)]

$$w(t) = -z(t) - pz(t-\tau) + q \int_{t-\sigma}^{t-\tau} z(s)\, ds.$$

Using the fact that $z(t)$ and $w(t)$ are solutions of (1), we obtain

$$w'(t) = qz(t-\tau).$$

We define m as in (18) and $\varphi(t) = e^{-\lambda t} z(t)$. We will show that $\lambda + \mu \in \Lambda(w)$ where $\mu = me^{\lambda_0 \tau}/((q/\lambda_0) + (-p)) > 0$. We have

$$\begin{aligned}
&-w'(t) + (\lambda+\mu)\, w(t) \\
&\quad = e^{\lambda t}\Big[-qe^{-\lambda\tau}\varphi(t-\tau) - (\lambda+\mu)\,\varphi(t) + (\lambda+\mu)(-p)\, e^{-\lambda\tau}\varphi(t-\tau) \\
&\qquad + (\lambda+\mu)\, qe^{-\lambda t} \int_{t-\sigma}^{t-\tau} e^{\lambda s}\varphi(s)\, ds \Big]
\end{aligned}$$

(and since $\varphi(t)$ is increasing and $\sigma > \tau$)

$$\begin{aligned}
&< e^{\lambda t}\varphi(t-\tau)\Big[-qe^{-\lambda\tau} - (\lambda+\mu) + (\lambda+\mu)(-p)\, e^{-\lambda\tau} \\
&\qquad + (\lambda+\mu)\, q\, \frac{e^{-\lambda t}}{\lambda} (e^{\lambda(t-\tau)} - e^{\lambda(t-\sigma)}) \Big] \\
&= e^{\lambda t}\varphi(t-\tau)\Big[-qe^{-\lambda\tau} - \lambda - \mu - \lambda pe^{-\lambda\tau} - \mu pe^{-\lambda\tau} \\
&\qquad + qe^{-\lambda\tau} - qe^{-\lambda\sigma} + \frac{\mu q}{\lambda} e^{-\lambda\tau} - \frac{\mu q}{\lambda} e^{-\lambda\sigma} \Big]
\end{aligned}$$

$$< e^{\lambda t}\varphi(t-\tau)\left[-\lambda - p\lambda e^{-\lambda\tau} - qe^{-\lambda\sigma} - \mu pe^{-\lambda\tau} + \frac{\mu q}{\lambda} e^{-\lambda\tau}\right]$$

$$\leqq e^{\lambda t}\varphi(t-\tau)\left[-m + \mu e^{-\lambda_0\tau}\left(\frac{q}{\lambda_0} + (-p)\right)\right] = 0$$

which implies that $\lambda + \mu \in \Lambda(w)$ and we are led to a contradiction.

The proof of the theorem is complete.

3. Remarks

In the papers [7, 8, 17] several sufficient conditions involving p, τ, q, and σ have been obtained under which all solutions of (1) oscillate. An advantage of working with these conditions rather than Eq. (2) directly is that the above conditions are explicit, while determining whether or not a real root to Eq. (2) exists may be quite a problem in itself. Nevertheless, in the present paper we can very easily derive sufficient conditions in terms of p, τ, q, and σ by comparing element of the set $\Lambda(z)$ in each case. For example, in the case where $-1 < p < 0$ we found that $q \in \Lambda(z)$ while $1/(-p)\,\tau$ is an upper bound of $\Lambda(z)$. Therefore if we assume that $q > 1/(-p)\,\tau$ we are led to a contradiction and every solution must oscillate. Summarizing we have the following:

Consider the neutral delay differential equation

$$x'(t) + px'(t-\tau) + qx(t-\sigma) = 0, \qquad t \geqq t_0 \tag{1}$$

where τ, q, σ are positive constants and p is a real parameter. Then each of the following conditions

$$(-p)\,\tau q > 1 \qquad \textit{when } -1 < p < 0, \tag{20}$$

$$\left(\frac{q}{1+p}\right)^3 \left(\frac{\sigma-\tau}{4}\right) > 1 \qquad \textit{when } p > 0, \tag{21}$$

$$\frac{q\tau}{p^2} > 1 \qquad \textit{when } p < -1 \textit{ and } \tau \geqq \sigma, \tag{22}$$

or

$$\frac{q\tau}{p^2 + (-p)\,q(\sigma-\tau)} > 1 \qquad \textit{when } p < -1 \textit{ and } \sigma > \tau \tag{23}$$

implies that every solution of (1) *oscillates.*

From the above result it follows that each of the conditions (20)–(23) implies that Eq. (2) has no real roots something that cannot be so easily determined by investigating directly the exponential equation (2).

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